TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 256, December 1979

REPARAMETRIZATION OF n-FLOWS OF ZERO ENTROPY

BY

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ABSTRACT. Let ϕ, ψ be two ergodic n-parameter flows which preserve finite probability measures on their spaces X, Y. Let T be a nullset-preserving map: $X \to Y$ sending each ϕ-orbit homeomorphically to a ϕ-orbit. Then ϕ, ψ are called homeomorphically orbit-equivalent. For $n = 1$, there has been developed a theory of such equivalence: "Loosely Bernoulli" theory. A completely parallel theory exists for higher dimensions, except that it is necessary to impose a certain natural "growth" restriction on T, a restriction which is vacuous in the case $n = 1$. In this paper we carry out this program, but only for the case of zero entropy.

1. Introduction. Let ϕ be an n-flow on the probability space (X, μ); by this we shall always mean that ϕ is a free, measure-preserving, ergodic action of $\mathbf{R}^n$ on (X, μ). By a *reparametrizing map* τ for ϕ we mean a jointly measurable τ: $X \times \mathbf{R}^n \to \mathbf{R}^n$ such that each $\tau(x, \cdot)$ is a homeomorphism: $\mathbf{R}^n \to \mathbf{R}^n$ and such that the function ϕ_τ: $(x, v) \mapsto \phi_{\tau(x,v)}(x)$ is again an n-flow on (X, μ_τ), where μ_τ is a certain probability measure equivalent to μ; ϕ_τ is then called a *reparametrization* of ϕ. If T: $(X, \mu_\tau) \to (Y, \nu)$ is a measure-isomorphism and $\psi_v = T\phi_{\tau(\cdot,v)}T^{-1}$, then T takes the measure class of μ to that of ν and sends orbits of ϕ homeomorphically to orbits of ψ: we call T a *homeomorphic orbit equivalence* between ϕ and ψ. Conversely, if T is a homeomorphic orbit equivalence linking ϕ and ψ, then $\psi_v T$ has the form $T\phi_{\tau(\cdot,v)}$, where τ is a reparametrizing map, and then $\psi = T\phi_\tau T^{-1}$. Thus, we may ask the question, "when is ψ homeomorphically orbit-equivalent to ϕ?" or, what is the same question, "when is ψ isomorphic to a reparametrization of ϕ?" This question was raised by Kakutani in 1942 **[K]**.

If $n = 1$, it is clear from Abramov's formula **[A]** that if ϕ and ψ are so related, then both lie in the same entropy class (zero, positive, or infinite). Furthermore, it was recently discovered that within each of these three entropy classes there exist flows which are *not* equivalent in this sense; this was shown in **[F1]**. In fact, there are uncountably many inequivalent flows, as was shown by Rudolph **[R1]**. In each entropy class there is a *simplest* equivalence class, the "loosely Bernoulli" class **[F1]**; this was also discovered by Katok **[Ka]** and Satayev **[S]**. The fact that the loosely Bernoulli (LB) flows form a single equivalence class was shown for zero entropy in **[Ka]** and for

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Received by the editors December 18, 1978.

AMS (MOS) subject classifications (1970). Primary 28A65.

positive entropy by Ornstein and Weiss in [W]. Following a suggestion of M. Ratner, we shall call LB flows of zero entropy *loosely Kronecker* flows.

If $n > 1$, the situation becomes more subtle: it has been shown by Rudolph in [**R2**] that *any* two n-flows may be linked by a homeomorphic orbit-equivalence when $n > 1$. This is a flow analogue to Dye's theorem [**D**]. However, the imposition of appropriate conditions on the homeomorphic orbit-equivalence T, conditions which are automatically satisfied in the one-dimensional case, give us a direct generalization of the 1-dimensional theory. Let $\tau(x, v)$ be defined by $T(\psi_v x) = \phi_{\tau(x,v)}(Tx)$. Then for $n = 1$ it may be seen that the condition $\int \|\tau(\cdot, v)\| \, d\mu < \text{const} \cdot \|v\|$ always holds. For $n > 1$, the imposition of such a condition for T and T^{-1} yields a familiar equivalence theory, as follows.

It is shown in [N] that the analogue of Abramov's formula holds under this assumption; and in the present paper, under the same assumption, we recapture the zero-entropy results of [**F1**], [**Ka**] and [W]. The analogous results in the case of positive entropy involve ideas beyond the scope of this paper, but will be carried out elsewhere.

Some of the results of this paper are contained in the second author's 1978 Ph.D. thesis in the Department of Mathematics at the University of California at Berkeley.

The first author wishes to thank for their support the Miller Institute for Basic Research in Science and the National Science Foundation (via Grant no. MCS-7505576 A01).

2. Reparametrization maps. It may be seen, with a little thought, that the following conditions (a), (b), (c) are necessary and sufficient for a measurable function $\tau: X \times \mathbf{R}^n \to \mathbf{R}^n$ to be a reparametrizing map for ϕ:

(a) $\tau(\cdot, v + w) = \tau(\cdot, v) + \tau(\phi_{\tau(\cdot,v)} \cdot, w)$ for all v and w;

(b) $\tau(x, \cdot)$ is, for a.e. x, a homeomorphism carrying Lebesgue measure to an equivalent measure.

(a) and (b) insure, in particular, that $(x, v) \mapsto \phi_{\tau(x,v)}x$ is a nonsingular n-flow on (X, μ).

Before describing the third condition, we make a brief digression.

Let $\Delta_\tau: X \times \mathbf{R}^n \to \mathbf{R}^n$ be a jointly measurable version of the Radon-Nikodym derivative of the image of Lebesgue measure with respect to Lebesgue measure. (Thus, where $\tau(x, \cdot)$ is sufficiently smooth, we have $\Delta_\tau(x, \cdot) = |\det \tau'(x, \cdot)|$.) It follows from (a) and the definition of Δ_τ that $\Delta_\tau(\phi_{\tau(x,v)}x, w) = \Delta_\tau(x, v + w)$ for a.e. (x, w), for every v. By translation invariance of Lebesgue measure, $\Delta_\tau(\phi_{\tau(x,v)}x, w - v) = \Delta_\tau(x, w)$ for a.e. (x, w), for every v. Therefore, for a.e. (x, w), the limit

$$\bar{\Delta}_\tau(x, w) = \lim_{N\to\infty} \frac{1}{|C_N|} \int_{C_N} \Delta_\tau(\phi_{\tau(x,v)}x, w - v) \, dv$$

exists and equals $\Delta_\tau(x, w)$. Here $C_N = [-N, N]^n$, and $|C_N|$ = Lebesgue measure of C_N. Let $A = \{(x, w)\colon \text{above limit exists}\}$. Then for each v, A is invariant under the map $(x, w) \mapsto (\phi_{\tau(x,v)}x, w - v)$, and $\bar{\Delta}_\tau(x, w) = \bar{\Delta}_\tau(\phi_{\tau(x,w)}x, w - v)$. Choose w_0 so $\{x\colon (x, w_0) \in A\}$ has full measure. Thus also $\{x\colon (\phi_{\tau(x,w_0)}x, 0) \in A\}$ has full measure. Applying the nonsingular map $x \mapsto \phi_{\tau(x,u-w_0)}x$ to this set, we see that $\{x\colon (\phi_{\tau(x,u)}x, 0) \in A\}$ has full measure for every u. Let $u = v + w$. Then it follows that $\bar{\Delta}_\tau(\phi_{\tau(x,v)}x, w) = \bar{\Delta}_\tau(x, w + v)$ for a.e. x. Drop the bar and call this version Δ_τ.

(c) $\int \Delta_\tau(\cdot, 0)\, d\mu < \infty$.

The reason for this is the following: $\Delta_\tau(\cdot, 0)\, d\mu$ is an invariant measure for ϕ_τ; this may be seen from a Rokhlin picture for ϕ, as described by Lind [L]. But we want this measure to be finite. Thus

$$d\mu_\tau = \left(\Delta_\tau(\cdot, 0)\Big/\int \Delta_\tau(\cdot, 0)\, d\mu\right) d\mu.$$

Reparametrizing maps may be composed by the rule $\sigma \circ \tau(\cdot, v) = \sigma(\cdot, \tau(\cdot, v))$; this is consistent with the way in which homeomorphic orbit-equivalences are composed. If τ is a reparametrizing map for ϕ, and σ for ϕ_τ, then $\sigma \circ \tau$ is one for ϕ, and $(\phi_\tau)_\sigma = \phi_{\sigma \circ \tau}$. In particular, τ has a unique inverse: a reparametrizing map τ^{-1} for ϕ_τ may be defined by setting $\tau^{-1}(x, \cdot)$ to be the inverse homeomorphism of $\tau(x, \cdot)$; thus $\tau^{-1} \circ \tau = \tau \circ \tau^{-1} =$ identity.

2.1. Definition. The reparametrizing map τ will be called *integrably Lipschitz* if $\int \|\tau(\cdot, v)\|\, d\mu \leqslant \text{const}\|v\|$ for all v and τ^{-1} satisfies the same condition. Note that an application of the cocycle identity then gives $\int \|\tau(\cdot, v) - \tau(\cdot, w)\|\, d\mu \leqslant \text{const} \cdot \|v - w\|$ with the same constant.

As remarked in the introduction, for $n \geqslant 2$ *any* two n-flows are homeomorphically orbit-equivalent. If, however, we restrict ourselves to integrably Lipschitz τ, we recapture the one-dimensional "loosely Bernoulli" theory. First, one has the analogue of Abramov's formula given in [N] which has the following corollary.

2.2. Theorem. *If τ is integrably Lipschitz, then ϕ and ϕ_τ have the same entropy class (zero, positive, infinite).*

Next, we describe a special case of another result of Rudolph [R3] which enables us to work with much nicer τ.

2.3. Definition. For any integrably Lipschitz τ, it turns out that $\int \tau^{-1}(\cdot, v)\, d\mu$ is a nonsingular linear transformation of R^n; we call the inverse of this transformaton $J(\tau)$.

2.4. Definition. A reparametrizing map τ will be called *tempered* if $\tau(x, \cdot)$ and $\tau^{-1}(x, \cdot)$ are C^∞ functions, and $\|\tau'(x, 0)\|$ and $\|\tau^{-1\prime}(x, 0)\|$ are bounded functions of x.

Then the case of Rudolph's result which interests us is:

2.5. THEOREM. *If τ is an integrably Lipschitz reparametrizing map for ϕ and $\varepsilon > 0$, then there exists a tempered reparametrizing map σ for ϕ with $\|\sigma'(\cdot, 0) - J(\tau)\|_\infty < \varepsilon$ and $\phi_\sigma \approx \phi_\tau$.*

Here is an intermediate condition on the reparametrizing map τ.

2.6. DEFINITION. τ will be called *uniformly Lipschitz* if $\|\tau(\cdot, v)\| \, \|v\|^{-1}$ is essentially bounded uniformly in $v \neq 0$, and likewise for τ^{-1}. As before, we note that $\|\tau(\cdot, v) - \tau(\cdot, w)\|/\|v - w\|$ will be essentially bounded for all pairs (v, w) with $v \neq w$, with the same bound as above: this is a consequence of the cocycle identity.

This will be useful in the proof of the equivalence theorem in §§5 and 6.

2.7. REMARK. If the n-flow ϕ has a factor $\bar{\phi}$, and $\bar{\tau}$ is a reparametrizing map for $\bar{\phi}$, then $\bar{\tau}$ "lifts" in an obvious way to a reparametrizing map τ for ϕ: if $x \mapsto \bar{x}$ is the quotient map which gives rise to $\bar{\phi}$ from ϕ, then set $\tau(x, v) = \bar{\tau}(\bar{x}, v)$. Clearly the properties in 2.1, 2.4 and 2.6 hold for τ if and only if they hold for $\bar{\tau}$.

3. Loosely Kronecker n-flows. Let D be a closed cell in $\mathbf{R}^n$ and let $|D|$ denote the Lebesgue measure of D. Let $\mathcal{P}$ be a finite set and δ the Kronecker function on $\mathcal{P} \times \mathcal{P}$. Let $\alpha, \beta \colon D \to \mathcal{P}$ be measurable. We recall:

3.1. DEFINITION. $d_D^{\mathcal{P}}(\alpha, \beta) = (1/|D|)\int_D \delta(\alpha(v), \beta(v))\, dv$. The superscript $\mathcal{P}$ will often be suppressed. Let $C_N = [-N, N]^n \subset \mathbf{R}^n$; for convenience we write d_N instead of d_{C_N}.

Let $\mathfrak{D}_D$ be the set of C^∞ self-diffeomorphisms of D which are the identity in a neighorhood of the boundary of D. Let $\|A\|$ denote the operator norm of the $n \times n$ matrix A when $\mathbf{R}^n$ is given the sup norm. For a matrix-valued function λ with domain D let $\|\lambda\|_\infty = \sup_{V \in D}\|\lambda(v)\|$.

3.2. DEFINITION. $f_D^{\mathcal{P}}(\alpha, \beta) = \inf_{h \in \mathfrak{D}_D}[d_D(\alpha \circ h, \beta) + \|h' - I\|_\infty]$. We take notational liberties with f similar to those described above. Note that d is a metric, but f is not. However, we *do* have

$$f_D(\alpha, \gamma) < f_D(\alpha, \beta) + (1 - f_D(\alpha, \beta))^{-1} f_D(\beta, \gamma).$$

Thus f_D gives the space of $(D, \mathcal{P})$ names a Hausdorff uniform structure. The approximate triangle inequality above will be sufficient for our purposes. Also, if $(\phi, \mathcal{P})$ is a process, then $f_D(x, y) = f_D(\mathcal{P}(x), \mathcal{P}(y))$, where $\mathcal{P}(x)(v) = \mathcal{P}(\phi_v(x)) = P$ if $\phi_v(x) \in P$.

3.3. DEFINITION. A process $(\phi, \mathcal{P})$ is *loosely Kronecker* (LK) if for all $\varepsilon > 0$ there exists an $M > 0$ such that if $N > M$ then there exists $E_N \subset X$ with $\mu(E_N) > 1 - \varepsilon$ and $f_N(x, y) < \varepsilon$ whenever $x, y \in E_N$. An n-flow (ϕ, μ) will be called *loosely Kronecker* if $(\phi, \mathcal{P})$ is LK for *all* partitions $\mathcal{P}$ of X.

A slight argument similar to the packing lemma in [N] shows that any LK n-flow must have zero entropy. Our terminology is motivated by the following example.

3.4. EXAMPLE. Let $X = T^{n+1}$, the $(n + 1)$-dimensional torus. Let $v \mapsto \bar{v}$ be a group-theoretic embedding of $\mathbf{R}^n$ into X as a dense subgroup and set $\phi_v x = x + \bar{v}$, where $+$ denotes the toral group operation. Let μ be Haar measure on X. Then ϕ is an n-flow on (X, μ); in fact, ϕ is LK.

To prove this, let $\varepsilon > 0$ and $\mathcal{P} = \{P_i\}_{i\in I}$ be a partition of X. Choose $\varepsilon_1 > 0$ to be specified later, open sets $U_i \supset P_i$ such that $\mu(\bigcup_{i\in I}(U_i \setminus P_i)) < \varepsilon_1$, and λ a translation-invariant metric on X. Let δ be a Lebesgue number for $\{U_i\}_{i\in I}$ with respect to λ. Choose N_1 so large that if $x,y \in X$, then $\lambda(x + \bar{v}, y) < \delta$ for some $v \in C_{N_1}$. Next, if N_2 is large enough, for all $v \in C_{N_1}$ there exists $h_{v,N} \in \mathfrak{D}_{C_N}$ if $N > N_2$ such that:

(1) $h_{v,N}(w) = v + w$ on $(1 - \varepsilon_1) \cdot C_N$;

(2) $\|h'_{v,N} - I\|_\infty < \varepsilon_1$.

We take N_2 also large enough to ensure

$$G_N = \left\{x \in X\colon |C_N|^{-1}\left|\left\{v \in C_N\colon c + \bar{v} \in \bigcup_{i\in I} (U_i \setminus P_i)\right\}\right| < \varepsilon_1\right\} \tag{3}$$

has measure at least $1 - \varepsilon_1$ if $N > N_2$.

Choose $x,y \in G_n$ and $v \in C_{N_1}$ such that $\lambda(x + \bar{v}, y) < \delta$. We use $h_{v,N}$ to match the $(\mathcal{P}, C_N)$ names of x and y. If $w \in (1 - \varepsilon_1)C_N$, then $x + \overline{h_{v,N}(w)}$ and $y + \bar{w}$ belong to the same element of the cover $\{U_i\}_{i\in I}$. If, further, w does not belong to the exceptional sets for x and y described in (3), we evidently have $\mathcal{P}(x + \overline{h_{v,N}(w)}) = \mathcal{P}(y + \bar{w})$. We thus have $f_N(x, y) < 1 - (1 - \varepsilon_1)^n + 3\varepsilon_1 + \varepsilon_1 < \varepsilon$ if ε_1 was chosen small enough. ϕ is therefore LK. $\square$

The property of being LK is stable under finite recodings:

3.5. LEMMA. *Let* $(\phi, \mathcal{P})$ *be an LK-process and* $V \in \mathbf{R}^n$ *be finite. Then* $(\phi, \mathcal{P}_V)$ *is again LK. (Here and elsewhere, by* $\mathcal{P}_V$ *we mean* $\bigvee_{v\in V} \phi_{-v}\mathcal{P}$.)

PROOF. Let $\varepsilon > 0$ and choose $\varepsilon_1 > 0$ to be specified later. Choose N_1 so large that: if $N > N_1$, then there exists $G_N \subset X$ with $\mu(G_N) > 1 - \varepsilon_1$ and $f_N^{\mathcal{P}}(x, y) < \varepsilon_1$, for $x,y \in G_N$; and if $N > N_1$, then $C'_N = \bigcap_{v\in V}(C_N - v)$ has Lebesgue measure at least $(1 - \varepsilon_1)|C_N|$. Let $N > N_1$, $x,y \in G_N$, and $h \in \mathfrak{D}_{C_N}$ with $\|h' - I\|_\infty < \varepsilon_1$ and $d_{C_N}(\mathcal{P}(x) \circ h, \mathcal{P}(y)) < \varepsilon_1$. Let

$$T = \{v \in C_N\colon \mathcal{P}(\phi_{h(v)}(x)) \neq \mathcal{P}(\phi_v(y))\}.$$

Then $|T| < \varepsilon_1|C_N|$, and since

$$\hat{T} = \{w \in C_N\colon \mathcal{P}_V(\phi_{h(w)}(x)) \neq \mathcal{P}_V(\phi_w(y))\} \subset (C_N \setminus C'_N) \cup \left(\bigcup_{v\in V} (T + v)\right),$$

we have that $|\hat{T}| < \varepsilon_1|C_N| + |V| \cdot |T| < (|V| + 1)\varepsilon_1|C_N|$. We see, then, that $f_N^{\mathcal{P}_V}(x, y) < (|V| + 2)\varepsilon_1 < \varepsilon$ if ε_1 was chosen small enough. $\square$

3.6. COROLLARY. *If the process* $(\phi, \mathcal{P})$ *is LK for some generator under the* n*-flow* ϕ*, then the* n*-flow* ϕ *is LK.*

3.7. PROPOSITION. *Let* $T \in GL(n, \mathbf{R})$ *and let* (ϕ, μ) *be LK. Let* $\tau(x, v) = T(v)$ *for* $(x, v) \in X \times \mathbf{R}^n$. *Then* (ϕ_τ, μ) *is LK.*

PROOF. Let $\{e_j\}_{j=1}^n$ be the standard basis of $\mathbf{R}^n$. Fix $\varepsilon > 0$, a partition $\{P_i\}_{i\in I}$ of X, and $\varepsilon_1 > 0$ to be specified later. By [P-S], there exists $v_1 \in \mathbf{R}^n$ such that ϕ_{v_1} is totaly ergodic, $\|v_1 - e_1\| < \varepsilon_1$, and $e_1 \in \{t_1 v_1 + \sum_{j=2}^n t_j e_j\colon 0 < t_j < 1\}$.[1] Choose N_1 so large that there exists $E_{N_1} \subset X$ with $\mu(E_{N_1}) > 1 - \varepsilon_1$ and $f_{N_1}(x, y) < \varepsilon_1$ for $x,y \in E_{N_1}$. Let $T(C_{N_1}) = D_N$. If we conjugate by T an f_{N_1}-matching, with error less than ε_1, of the $\mathcal{P}$-names of x and y in ϕ, we obtain an $f_{D_{N_1}}$-matching, with error less than $(\|T\|\,\|T^{-1}\| + 1)\varepsilon_1$, of the $\mathcal{P}$-names of x and y in ϕ_τ.

Let $\hat{C} = \{t_1 v_1 + \sum_{j=2}^n t_j e_j\colon t_j \in [-N_1, N_1]\}$, $\hat{D} = T(\hat{C})$, and

$$L_N = \left\{ v = \sum_{j=2}^n m_j e_j + m_1 v_1\colon m_j \in 2N_1\mathbf{Z}, \hat{D} + T(v) \subset C_N \right\}.$$

If N_2 is sufficiently large, for all $N > N_2$ we will have: (1) $|\bigcup_{v\in L_N}(\hat{D} + T(v))| > (1 - \varepsilon_1)|C_N|$; (2) for some $F_N \subset X$ with $\mu(F_N) > 1 - \varepsilon_1$,

$$\left|\{v \in L_N\colon \phi_v(x) \in E_{N_1}\}\right| > (1 - \varepsilon_1)|L_N|$$

for all $x \in F_N$. It is clear that (1) can be satisfied by choice of N_2 large enough; that (2) can also be, is a consequence of the ergodicity of $\phi_{N_1 v_1}$.

Note that $D_{N_1} \subset \hat{D}$ and $|D_{N_1}| > (1 - \varepsilon_1)|\hat{D}|$. For $x,y \in F_N$, let $G_{x,y} = \{v \in L_N\colon \phi_v(x), \phi_v(y) \in E_{N_1}\}$. If $v \in G_{x,y}$ then some $h_v \in \mathfrak{D}_{D_{N_1}}$ matches the $(\mathcal{P}, D_{N_1})$-names in the flow ϕ_τ of $\phi_{T(v)}(x)$ and $\phi_{T(v)}(y)$ with error less than $(\|T\|\,\|T^{-1}\| + 1)\varepsilon_1$. These h_v's paste together smoothly because each is the identity in a neighborhood of the boundary of its domain; by defining the match to be the identity throughout the rest of C_N, a calculation shows that the error over all of C_N is at most

$$1 - (1 - 2\varepsilon_1)(1 - \varepsilon_1) + 2(\|T\|\,\|T^{-1}\| + 1)\varepsilon_1 < \varepsilon$$

if ε_1 is small enough. □

In fact, much more is true; any integrably Lipschitz reparametrization of an LK n-flow is again LK. For the proof of this, we need a technical lemma whose proof may be safely omitted.

3.8. LEMMA. *For all sufficiently small* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any* C^1 *embedding* $h\colon (1 - \delta)C_N \to C_N$ *with* $\|h' - I\|_\infty < \delta$, *there exists* $h_1 \in \mathfrak{D}_{C_N}$ *with* $\|h_1' - I\|_\infty < \delta$ *and* $h_1(v) = h(v)$ *if* $v \in (1 - \varepsilon)C_N$.

3.9. THEOREM. *Let the n-flow* ϕ *on* (X, μ) *be carried by the integrably Lipschitz reparametrizing map* τ *to* ϕ_τ, *an n-flow on* (X, μ_τ). *If* ϕ *is LK, so is* ϕ_τ.

[1]The referee has pointed out that, by use of a standard "averaging" argument, this use of the Pugh-Shub result could be avoided.

PROOF. By following τ with a linear reparametrizing map we may assume $J(\tau) = I$. Let $\mathcal{P}$ be a partition of X, $\varepsilon > 0$, and $\bar{\varepsilon} > 0$ to be determined later. By Theorem 2.5, there exists $\bar{\tau}$, a tempered reparametrizing map for ϕ, such that (1) $\|\bar{\tau}'(x, 0) - I\|$, $\|\bar{\tau}'(x, 0)^{-1} - I\| < \bar{\varepsilon}$ for a.e. $x \in X$; and (2) $\phi_\tau \approx \phi_{\bar{\tau}}$ via some isomorphism θ. Let $\overline{\mathcal{P}}$ be the partition to which $\mathcal{P}$ is carried under isomorphism from ϕ_τ to $\phi_{\bar{\tau}}$. Let $\tau_x(u) = \tau(x, v)$ for $x \in X$, $v \in \mathbf{R}^n$.

Note that if $x, y \in X$, $h \in \mathfrak{D}_{C_N}$ such that $\|h' - I\|_\infty < \bar{\varepsilon}$, and $h_1 = \tau_x \circ h \circ \tau_y^{-1}\colon \tau_y(C_N) \to \tau_x(C_N)$, then (3) $h_1((1 - 3\bar{\varepsilon}) \cdot C_N) \subset C_N$ and (4) $\|h_1' - I\| < 3(\bar{\varepsilon} + \bar{\varepsilon}^2) + \bar{\varepsilon}^3$.

Now choose N_1 so large that if $N \geqslant N_1$, then for all $x, y \in E_N \subset X$ with $\mu(E_N) > 1 - \bar{\varepsilon}$ we have that $f_N^{\mathcal{P}}(x, y) < \bar{\varepsilon}$ (in the original flow ϕ). Fix $x, y \in E_N$ and $h \in \mathfrak{D}_{C_N}$ an f_N-matching of the $\overline{\mathcal{P}}$-names of x and y with error less than $\bar{\varepsilon}$. Defining h_1 as above, by (3), (4), and Lemma 3.8 we will have, whenever $\bar{\varepsilon}$ is small enough, an $h_2 \in \mathfrak{D}_{C_N}$ such that (5) $\|h_2' - I\|_\infty < \varepsilon/3$ and (6) $|C_N|^{-1}|\{v \in C_N\colon h_1(v) = h_2(v)\}| > 1 - \varepsilon/3$.

If $\bar{\varepsilon}$ is perhaps smaller still, it follows from (5), (6), and the fact that h_1 matches the $(\overline{\mathcal{P}}, \tau_x(C_N))$ name of x and the $(\overline{\mathcal{P}}, \tau_y(C_N))$ names of y (in the flow $\phi_{\bar{\tau}}$) with error less than $\bar{\varepsilon}$ that h_2 matches the $(\overline{\mathcal{P}}, C_N)$ names of x and y in $\phi_{\bar{\tau}}$ with error less than ε. Under the isomorphism this pulls back to an f_N-matching of the $\mathcal{P}$-names of $\theta^{-1}(x)$ and $\theta^{-1}(y)$ in ϕ_τ with the same error, and if $\bar{\varepsilon}$ is also so small that $\mu_\tau(E_N) > 1 - \varepsilon$, we are finished. □

4. Finitely fixed zero entropy n-flows. For more detailed informaton about the metrics $\overline{d_N}$ and $\bar{d}$ we refer the reader to [O].

4.1. DEFINITION. By a *joining* of (X, μ) and (Y, ν) we mean a measure ρ on $X \times Y$ which has μ and ν as its marginals. If ϕ is an action of $\mathbf{R}^n$ on (X, μ), ψ an action on (Y, ν), and $\mathcal{P}$ and $\mathcal{Q}$ are corresponding indexed partitions of X and Y, then by $\overline{d_N}((\phi, \mathcal{P}), (\psi, \mathcal{Q}))$ we mean the infimum

$$\int d_N(\mathcal{P}(x), \mathcal{Q}(y))\, d\rho(x, y) \tag{$*$}$$

over all joinings $(X \times Y, \rho)$. Observing that $\overline{d_N}$ increases with N, we set $\bar{d}((\phi, \mathcal{P}), (\psi, \mathcal{Q})) = \lim_{N \to \infty} \overline{d_N}((\phi, \mathcal{P}), (\psi, \mathcal{Q}))$.

4.2. DEFINITION. $\overline{f_N}$ and $\bar{f}$ are defined analogously, except that $(*)$ is replaced by

$$\int f_N(\mathcal{P}(x), \mathcal{Q}(y))\, d\rho(x, y).$$

It is clear that $\overline{f_N} \leqslant \overline{d_N}$ for each N. However:

4.3. PROPOSITION. *Given $(\phi, \mathcal{P})$, N, and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\overline{f_N}((\phi, \mathcal{P}), (\psi, \mathcal{Q})) < \delta$, then $\overline{d_N}((\phi, \mathcal{P}), (\psi, \mathcal{Q})) < \varepsilon$.*

PROOF. It follows from the Lebesgue continuity theorem that if $f_N(\mathcal{P}(x), \mathcal{Q}(y)) < \delta(x)$, then $d_N(\mathcal{P}(x), \mathcal{Q}(y)) < \varepsilon$. The proposition follows. $\square$

4.4. DEFINITION. Let $(\phi, \mathcal{P})$ be of entropy zero. We say $(\phi, \mathcal{P})$ is *finitely fixed* (FF) if, given $\varepsilon > 0$, there exist N and $\delta > 0$ such that if $\overline{f_N}((\phi, \mathcal{P}), (\psi, \mathcal{Q})) < \delta$, then $\bar{f}((\phi, \mathcal{P}), (\psi, \mathcal{Q})) < \varepsilon$.

4.5. THEOREM. *If $(\phi, \mathcal{P})$ is LK, then it is FF.*

PROOF. Choose $\varepsilon > 0$. Choose N such that there is a set $A \subset X$ with $\mu(A) > 1 - \varepsilon/8$ so that if $x_1, x_2 \in A$, then $f_N(\mathcal{P}(x_1), \mathcal{P}(x_2)) < \varepsilon/8$. By making $\overline{f_N}((\phi, \mathcal{P}), (\psi, \mathcal{Q}))$ small enough, we can guarantee that there is a set $B \subset \overline{Y}$ with $\nu(B) > 1 - \varepsilon/8$, and for each $y \in B$ some $x \in A$ with $f_N(\mathcal{P}(x), \mathcal{Q}(y)) < \varepsilon/8$. Then for *all* $x \in A$ and $y \in B$ we get $f_N(\mathcal{P}(x), \mathcal{Q}(y)) < \varepsilon/3$ by the modified triangle inequality for f, whenever $\varepsilon < 1$.

Now choose M so large that by the ergodic theorem there exists a set $A_0 \subset X$ with $\mu(A_0) > 1 - \varepsilon/2$ such that if $x \in A_0$, then for a fraction at least $1 - \varepsilon/4$ of the v in $N\mathbf{Z}^n \cap C_M$, we have $\phi_v(x) \in A$; and similarly, there is a set $B_0 \subset Y$ with $\nu(B_0) > 1 - \varepsilon/2$ so that if $y \in B_0$, then for a fraction at least $1 - \varepsilon/4$ of the v in $N\mathbf{Z}^n \cap C_M$, we have $\psi_v(y) \in B$. Then for $x \in A_0$ and $y \in B_0$ we have $f_M(\mathcal{P}(x), \mathcal{Q}(y)) < \varepsilon/2 + \varepsilon/4 + \varepsilon/3 +$ another $\varepsilon/6$ to take care of edge effects in case N does not divide M. Thus, for *any* joining ρ of $(\phi, \mathcal{P})$ and $(\psi, \mathcal{Q})$, we have

$$\int f_M(\mathcal{P}(x), \mathcal{Q}(y))\, d\rho(x,y) < \varepsilon + 1 - \rho(A_0 \times B_0)$$
$$< \varepsilon + \rho((x \setminus A_0) \times Y) + \rho(X \times (Y \setminus B_0)) < 2\varepsilon. \quad \square$$

4.6. PROPOSITION. *If $(\phi, \mathcal{P})$ is FF, then $(\phi, \mathcal{P})$ is LK.*

PROOF. Fix $\varepsilon > 0$. Choose N and δ for $\varepsilon/4$ in the FF definition. Take M much larger than N and choose any point x whose $(\mathcal{P}, C_M)$ name has δ-good empirical distribution of $(\mathcal{P}, C_N)$ names. Now make a periodic process $(\bar{\phi}, \overline{\mathcal{P}})$ by using the $(\mathcal{P}, C_M)$ name of x. This process is within δ of $(\phi, \mathcal{P})$ in its distribution of $(\mathcal{P}, C_N)$ names; consequently $\bar{f}((\phi, \mathcal{P}), (\bar{\phi}, \overline{\mathcal{P}})) < \varepsilon/4$. Now, the periodic process $(\bar{\phi}, \overline{\mathcal{P}})$ is certainly LK, for if L is large, any two C_{ML} names can be matched well by an $h \in \mathfrak{D}_{C_{ML}}$ which translates a large portion of C_{ML} by some vector of length at most $2M$. Take an $\bar{f}$-joining of $(\phi, \mathcal{P})$ with $(\bar{\phi}, \overline{\mathcal{P}})$ which has gap less than $\varepsilon/4$. Then there is a set $A \subset X$ of measure at least $1 - \varepsilon/4$ such that the $(\mathcal{P}, C_{ML})$ name of every $x \in A$ matches with some $(\overline{\mathcal{P}}, C_{ML})$ name with f-error less than $\varepsilon/4$. Since any two $(\overline{\mathcal{P}}, C_{ML})$ names are close, it follows that any two $(\mathcal{P}, C_{ML})$ names of points in A are close for f by the modified triangle inequality. $\square$

5. The "Sinai Theorem" for zero entropy FF n-flow. We prove now the homeomorphic orbit-equivalence analogue of Ornstein's [O] version of Sinai's theorem [S].

5.1. THEOREM. *Let ψ be an n-flow on (Y, ν), and let $(\phi, \mathcal{P})$ be FF of entropy zero on (X, μ). Suppose $\mathcal{R}$ is a partition of Y with $\bar{f}((\phi, \mathcal{P}), (\psi, \mathcal{R})) < \varepsilon/100$. Then there is a uniformly Lipschitz reparametrizing map τ for ψ and a partition $\mathcal{R}'$ of Y such that*: (1) $\|\tau(\cdot, v) - v\|, \|\tau^{-1}(\cdot, v) - v\| < \varepsilon\|v\|$ *for a.e. x for all* v, (2) $|\mathcal{R} - \mathcal{R}'| < \varepsilon$, *and* (3) $(\psi_\tau, \mathcal{R}') \approx (\phi, \mathcal{P})$.

5.2. REMARK. Sharper results may be obtained. For example, it can be arranged to have τ tempered, with $\|\tau'(\cdot, 0) - I\|_\infty, \|\tau'(\cdot, 0)^{-1} - I\| < \varepsilon$. However, we satisfy ourselves with this simplest version. The result will follow easily from

5.3. FUNDAMENTAL LEMMA. *Same as Theorem* 5.1, *but choose* $\varepsilon' > 0$ *and replace* (3) *by* (3)$'$:

$$\bar{f}((\psi_\tau, \mathcal{R}'), (\phi, \mathcal{P})) < \varepsilon'.$$

PROOF OF 5.3. Let N' and δ' be the quantities required for $(\phi, \mathcal{P})$ and ε' in the definition of FF. Then we need to find $\mathcal{R}'$ and τ satisfing (1), (2), and (3)$''$: $\overline{f_{N'}}((\psi_\tau, \mathcal{R}'), (\phi, \mathcal{P})) < \delta'$.

Since $\bar{f}((\phi, \mathcal{P}), (\psi, \mathcal{R})) < \varepsilon/100$, we can choose for each N a measure ρ_N on $X \times Y$ with marginals μ and ν such that ρ_N gives measure less than $\varepsilon/100$ to those (x, y) for which $f_N(\mathcal{P}(x), \mathcal{Q}(y)) > \varepsilon/100$. Now, for given $\delta > 0$ (to be chosen later), choose N large enough that the empirical distribution of $(\mathcal{P}, C_{N'})$ names in the $(\mathcal{P}, C_N)$ name of x lies within δ of the true distribution, in the $\bar{d}$ metric, except for a set of x of measure less than δ. This is possible because of the ergodic theorem for ϕ. Let S be the set of nonexceptional x, so that $\mu(S) > 1 - \delta$.

Now let $\mathcal{B}$ be a disjoint collection of $\mathcal{R}_{C_N}$-measurable sets each having $d^{\mathcal{P}}_{C_N}$ diameter less than $\varepsilon/100(1 + \varepsilon/100)^n$, with $\gamma(\bigcup_{B\in\mathcal{B}} B) > 1 - \varepsilon/100$. Such a $\mathcal{B}$ certainly exists, since we do not care how big $|\mathcal{B}|$ is. Consider the total measure of those $B \in \mathcal{B}$ which contain points y such that for some $x \in S$, $f_N(\mathcal{P}(x), \mathcal{R}(y)) < \varepsilon/100$. This total measure is then at least $1 - \varepsilon/50$ if $\delta < \varepsilon/100$. Choose y_B with the above property for each such B and a corresponding $x_B \in S$. Thus for each such B we have some $h_B \in \mathcal{D}_{C_N}$ with $\|h'_B - I\|_\infty$, $\|h_B'^{-1} - I\|_\infty < \varepsilon/100$ and $d_N(\mathcal{R}(y_B) \circ h_B, \mathcal{P}(x_B)) < \varepsilon/100$. Now, for any $y \in B$ we have $d_N(\mathcal{R}(y), \mathcal{R}(y_B)) < \varepsilon/(1 + \varepsilon/100)^n \cdot 100$, so $d_N(\mathcal{R}(y) \circ h_B, \mathcal{R}(y_B) \circ h_B) < \varepsilon/100$ in view of the bound on $\|h'_B - I\|_\infty$, and $d_N(\mathcal{P}(x_B), \mathcal{R}(y) \circ h_B) < \varepsilon/50$ for all $y \in B$.

Now build a Rokhlin tower $E = C_N F$ in Y with error less than δ. This may be chosen to be a "strong" Rokhlin tower in the following sense. Set

$\tilde{\mathfrak{R}}_v = \psi_v^{-1}(\mathfrak{R}|\psi_v(F))$ for $v \in C_N$; put a measure $\tilde{\nu}$ on F by normalizing the projection of ν on E. Then the tower may be chosen so that the $\bar{d}$ distances between the processes $\{\tilde{\mathfrak{R}}_v: v \in C_N\}$ and $\{\psi_v(\mathfrak{R}): v \in C_N\}$ is less than δ (see [F2]).

Now copy $\mathfrak{B}$ into F, using the joining of $\{\tilde{\mathfrak{R}}_v: v \in C_N\}$ and $\{\psi_v(\mathfrak{R}): v \in C_N\}$. Call the new family $\tilde{\mathfrak{B}}$. If δ has been chosen small enough, then for "most" $\tilde{B} \in \tilde{\mathfrak{B}}$ we have an associated point $x_B \in S$ such that for "most" $y \in \tilde{B}$ we have

$$d\big(\{\mathfrak{R}_{h_B(v)}(y): v \in C_N\}, \mathfrak{P}_{C_N}(x_B)\big) < \varepsilon/40.$$

The set of y not included in such a $\tilde{B}$, or for which the above inequality fails, may be assumed to have total measure less than $\varepsilon/40$ by choice of δ sufficiently small. Let T be the "good" y's.

Now we define $\mathfrak{R}'$. If $y = \psi_v(y_0)$, $y_0 \in \tilde{B}$, then set $\mathfrak{R}'(y) = \mathfrak{P}(\phi_{h_B^{-1}(v)}(x_B))$. For $y = \psi_v(y_0)$ with $y_0 \in F \setminus \bigcup_{\tilde{B} \in \tilde{\mathfrak{B}}} \tilde{B}$, choose some fixed $\bar{x} \in S$ and set always $\mathfrak{R}'(y) = \mathfrak{P}(\phi_v(\bar{x}))$. Thus $\mathfrak{R}'|E$ is defined. Outside E $\mathfrak{R}'$ may be defined arbitrarily.

The reparametrizing map τ will now be defined. If $y = \psi_v(y_0)$, $y_0 \in B$, set $\theta(y) = \psi_{h_B(v)}(y)$; if $y = \psi_v(y_0)$, $y_0 \in F \setminus \bigcup_{\tilde{B} \in \tilde{\mathfrak{B}}} \tilde{B}$ or if $y \notin E$, set $\theta(y) = y$. Then τ is defined by taking $\tau(y, v) = w$ if $\theta(\psi_v(y)) = \psi_w(y)$. Since each h_B is the identity in a neighborhood of the boundary of C_N, it follows that τ is C^∞ on orbits. Furthermore $\|\tau'(y, v) - I\|$, $\|\tau'(y, v)^{-1} - I\| < \varepsilon/100$.

Since $d(\{\tilde{\mathfrak{R}}_{h_B(v)}(y): v \in C_N\}, \mathfrak{P}_{C_N}(x_B)) < \varepsilon/40$ when $y \in \tilde{B} \cap T$, and since $\mathfrak{R}'(\psi_{h_B(u)}(y)) = \mathfrak{P}(\phi_v(x_B))$ for such y, it follows that $|\mathfrak{R}' - \mathfrak{R}| < \varepsilon$.

It remains to show that $\overline{f_{N'}}((\psi_\tau, \mathfrak{R}'), (\phi, \mathfrak{P})) < \delta'$. Now, the $(\mathfrak{R}', C_N)$ name under ψ_τ of a point y in F is the $(\mathfrak{P}, C_N)$ name under ϕ of some point in S; consequently the empirical distribution of the $(\mathfrak{R}', C_{N'})$ names under ψ_τ within the $(\mathfrak{R}', C_N)$ name of y under ψ_τ is within δ of the distribution of $(\mathfrak{P}, C_N)$ names. If N is much larger than N' and δ is sufficiently small (recalling that $\nu(Y \setminus E) < \delta$), we have the desired result.

Completion of Proof of 5.1. In [O], or [F2], one simply makes a sequence of successively better partitions which converges. Here one must also make a sequence of reparametrizing maps $\tau_1, \tau_2, \ldots$ so that τ_1, $\tau_2 \circ \tau_1$, $\tau_3 \circ \tau_2 \circ \tau_1, \ldots$ converge to a reparametrizing map τ satisfying the uniform Lipschitz condition (1).

Suppose $\|\tau_j(\cdot, v) - v\|_\infty < \delta_j\|v\|$. Set $\sigma_0 = \text{ident}$, $\sigma_{j+1} = \tau_{j+1} \circ \sigma_j$. Then

$$\|\sigma_{j+1}(\cdot, v) - \sigma_j(\cdot, v)\|_\infty < \delta_{j+1}\|\sigma_j(\cdot, v)\|_\infty < \delta_{j+1}\prod_1^j (1 + \delta_j)\|v\|,$$

If $\sum_{j=1}^\infty \delta_j < \infty$, then $\Pi(1 + \delta_j) < \infty$ and we have that

$$\sum_{j=0}^\infty \|\sigma_{j+1}(\cdot, v) - \sigma_j(\cdot, v)\|_\infty < \left(\prod_{j=1}^\infty (1 + \delta_j)\right)\sum_{j=1}^\infty \delta_j\|v\|.$$

Thus $\sigma_j(\cdot, v)$ converges in $L_\infty(\mu)$. We may choose the δ_j's so small that $(\prod_{j=1}^\infty(1+\delta_j))\sum_{j=1}^\infty \delta_j < \varepsilon$, and then the limiting τ will satisfy $\|\tau(\cdot, v) - v\|_\infty < \varepsilon\|v\|$.

Since we also have $\|\tau_j^{-1}(\cdot, v) - v\|_\infty < \delta_j\|v\|$, it follows that the $\sigma_j^{-1} = \tau_1^{-1} \circ \cdots \circ \tau_j^{-1}$ also converge, and since their limit $\bar\tau$ satisfies $\bar\tau \circ \tau = \tau \circ \bar\tau$ = identity, we have that $\bar\tau = \tau^{-1}$ and so $\tau(x, \cdot)$ is for a.e. x a homeomorphism.

Since $\tau(x, \cdot)$ is for a.e. x Lipschitz in both directions, it carries Lebesgue measure to an equivalent measure; and because of the uniformity of the Lipschitz constants, the Radon-Nikodym derivatives Δ_τ and $\Delta_{\tau^{-1}}$ are uniformly bounded, and it follows that τ is a uniformly Lipschitz reparametrization map for ψ.

Now we have partitions $\mathcal{R}_j \to \mathcal{R}$ and $\bar f((\psi_{\sigma_j}, \mathcal{R}_j), (\phi, \mathcal{P})) < \delta_j$. It follows easily that $\bar f((\psi_\tau, \mathcal{R}), (\phi, \mathcal{P})) = 0$, so that $(\psi_\tau, \mathcal{R}') \approx (\phi, \mathcal{P})$. $\square$

A more careful application of the fundamental lemma gives:

5.4. COROLLARY. *Suppose $\mathcal{P}$ is a generator under ϕ and $(\phi, \mathcal{P})$ is FF of entropy zero. Then given $\varepsilon > 0$ there is some δ and N such that if $\mathcal{R}$ is a partition (Y, ν) satisfying $\bar f_N((\psi, \mathcal{R}), (\phi, \mathcal{P})) < \delta$, then there is a reparametrization ψ_τ of ψ on (Y, ν_τ) and another partition $\mathcal{R}'$ with:* (1) $\|\tau(\cdot, v) - v\|_\infty$, $\|\tau^{-1}(\cdot, v) - v\| < \varepsilon$, (2) $|\mathcal{R} - \mathcal{R}'| < \varepsilon$, *and* (3) $(\psi', \mathcal{R}') \approx (\phi, \mathcal{P})$.

6. The equivalence theorem for FF zero entropy n-flows.

6.1. THEOREM. *Suppose $\mathcal{P}$ is a generator for the zero entropy n-flow ϕ and $(\phi, \mathcal{P})$ is FF. Suppose $(\psi, \mathcal{Q})$ satisfies the same assumptions. Then given $1 > \varepsilon > 0$ there is a uniformly Lipschitz reparametrizing map τ for ϕ with $\|\tau(\cdot, v) - v\|_\infty$, $\|\tau^{-1}(\cdot, v) - v\|_\infty < \varepsilon\|v\|$ such that $\phi_\tau \approx \psi$.*

The proof is based on the following lemma, which is almost Lemma 5.11 of [F2].

6.2. LEMMA. *Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions with $(\phi, \mathcal{P})$ and $(\phi, \mathcal{Q})$ of zero entropy and FF. Let $\mathcal{P}_{\mathbf{R}^n} \supset^{\varepsilon/10} \mathcal{Q}$. Then there is a reparametrizing map τ for ϕ and a partition $\bar{\mathcal{Q}}$ such that, setting $\bar\phi = \phi_\tau$, we have:* (1) $\|\tau(\cdot, v) - v\|_\infty$, $\|\tau^{-1}(\cdot, v) - v\|_\infty < \varepsilon\|v\|$, (2) $|\bar{\mathcal{Q}} - \mathcal{Q}| < \varepsilon$, (3) $(\bar\phi, \bar{\mathcal{Q}}) \approx (\phi, \mathcal{Q})$, *and* (4) $\mathcal{P} \subset^\varepsilon \bigvee_{v \in \mathbf{R}^n} \bar\phi_v^{-1}(\bar{\mathcal{Q}})$.

PROOF. Choose a finite set $V \subset \mathbf{R}^n$ so that $\mathcal{P}_V \supset \mathcal{Q}_0$ with $|\mathcal{Q}_0 - \mathcal{Q}| < \varepsilon/8$. Apply Corollary 5.4 to $(\phi, \mathcal{P}_V)$ (which is clearly FF because $(\phi, \mathcal{P})$ is) to get δ and N for $\varepsilon/4$. Now build a partition $\mathcal{P}' \subset \mathcal{Q}_{\mathbf{R}^n}$ so that $(\phi, (\mathcal{P} \vee \mathcal{Q})_V)$ and $(\phi, (\mathcal{P}' \vee \mathcal{Q})_V)$ match within δ on C_K, where $K > N$ and $V \subset C_K$, and so that also $(\mathcal{P} \vee \mathcal{Q})_V$ and $(\mathcal{P}' \vee \mathcal{Q})_V$ have distribution distance so small that if we set $\mathcal{Q}_0'$ equal to the partition in $\mathcal{P}_V'$ built like $\mathcal{Q}_0$ in $\mathcal{P}_V$, then $|\mathcal{Q}_0' - \mathcal{Q}| < |\mathcal{Q}_0 - \mathcal{Q}| + \varepsilon/8$. Thus $\mathcal{Q}_0' \subset \mathcal{P}_V'$ and $|\mathcal{Q}_0' - \mathcal{Q}| < \varepsilon/4$. All this can be done by Lemma 5.10 of [F2].

Now since $\bar{d}_{C_K}((\phi, \mathcal{P}_V), (\phi, \mathcal{P}_V')) < \delta$, there is some reparametrizing map σ for $\phi|\mathcal{Q}_{\mathbf{R}^n}$, lifting to a reparametrizing map $\bar{\sigma}$ for ϕ, and some $\overline{\mathcal{P}} \subset \mathcal{Q}_{\mathbf{R}^n}$, with (1) $\|\sigma(\cdot, v) - v\|_\infty$, $\|\sigma^{-1}(\cdot, v) - v\|_\infty < \varepsilon\|v\|/4$, (2) $|\overline{\mathcal{P}}_V - \mathcal{P}_V'| < \varepsilon/4$, and (3) $(\phi, \mathcal{P}) \approx (\psi, \overline{\mathcal{P}})$, where $\psi = \phi_\sigma$.

Set $\overline{\mathcal{P}}_V^\psi = \bigvee_{v\in V} \psi_V^{-1}(\overline{\mathcal{P}})$ and let $\overline{\mathcal{Q}_0}$ be constructed from $\overline{\mathcal{P}}_V^\psi$ in the same way in which $\mathcal{Q}_0$ and $\mathcal{Q}_0'$ are constructed from $\mathcal{P}_V$ and $\mathcal{P}_V'$. Then $\overline{\mathcal{Q}_0}$ is also the image of $\mathcal{Q}_0$ under the isomorphism from $(\phi, \mathcal{P})$ to $(\psi, \overline{\mathcal{P}})$. Also clearly $|\overline{\mathcal{Q}_0} - \mathcal{Q}_0'| < \varepsilon/4$.

Now reverse roles. Choose a finite set $W \subset \mathbf{R}^n$ so that $\mathcal{Q}_W \supset \overline{\mathcal{P}_0}$ with $|\overline{\mathcal{P}_0} - \overline{\mathcal{P}}| < \varepsilon/8$.

We will apply Corollary 5.4 to $(\psi, \mathcal{Q}_W^\psi)$ this time (which is FF by Proposition 4.6, Lemma 3.5 and Proposition 4.5) for $\varepsilon/4$; call the relevant quantities δ and N again, since the old ones will not be used any more.

Apply Lemma 5.10 of [F2]: choose L so that $W \subset C_L$ and $L > N$; build $\mathcal{Q}' \subset \overline{\mathcal{P}}_{\mathbf{R}^n}^\psi$ so that $(\psi, (\overline{\mathcal{P}} \vee \mathcal{Q})_W^\psi)$ and $(\phi, (\mathcal{P} \vee \mathcal{Q}')_W)$ have a better than δ match on C_L, while $(\overline{\mathcal{P}} \vee \mathcal{Q})_W^\psi$ and $(\mathcal{P} \vee \mathcal{Q}')_W$ are so close in joint distribution that if $\overline{\mathcal{P}}_0' \subset \mathcal{Q}_W'$ corresponds to $\mathcal{P}_0' \subset \mathcal{Q}_W^\psi$, then $|\overline{\mathcal{P}}_0' - \overline{\mathcal{P}_0}| < \varepsilon/2$; further, the partitions $\mathcal{Q}_0 \vee \mathcal{Q}'$ in $\mathcal{P}_V \vee \mathcal{Q}'$ and the corresponding $\overline{\mathcal{Q}_0} \vee \mathcal{Q}$ in $\overline{\mathcal{P}}_V^\psi \vee \mathcal{Q}$ have such close joint distribution that $|\overline{\mathcal{Q}}_0 - \mathcal{Q}'| < |\mathcal{Q}_0 - \mathcal{Q}| + \varepsilon/8 < \varepsilon/4$.

As before, we have some reparametrizing map ρ for ψ and some $\bar{\mathcal{Q}} \subset \overline{\mathcal{P}}_{\mathbf{R}^n}$ with (1) $\|\rho(\cdot, v) - v\|_\infty$ and $\|\rho^{-1}(\cdot, v) - v\|_\infty < \varepsilon/4\|v\|$, (2) $|\bar{\mathcal{Q}}_W - \mathcal{Q}_W'| < \varepsilon/4$, and (3) $(\psi, \mathcal{Q}) \approx (\bar{\phi}, \bar{\mathcal{Q}})$, where $\bar{\phi} = \psi_\rho$. Thus the partition $\overline{\mathcal{P}}_0 \subset \bar{\mathcal{Q}}_W^{\bar\phi}$ corresponding to $\mathcal{P}_0' \subset \mathcal{Q}_W$ and $\overline{\mathcal{P}} \subset \mathcal{Q}_W^\psi$ must satisfy $|\overline{\mathcal{P}}_0 - \mathcal{P}_0'| < \varepsilon/2$.

Now $\bar{\mathcal{Q}}_{\mathbf{R}^n}^{\bar\phi} \supset \bar{\mathcal{Q}}_W^{\bar\phi} \supset \overline{\mathcal{P}}_0$, and $|\overline{\mathcal{P}}_0 - \mathcal{P}_0'| < \varepsilon/2$, and $|\overline{\mathcal{P}}_0' - \overline{\mathcal{P}}| < \varepsilon/2$. So $\bar{\mathcal{Q}}_{\mathbf{R}^n}^{\bar\phi} \supset^\varepsilon \overline{\mathcal{P}}$.

Similarly, $|\mathcal{Q} - \bar{\mathcal{Q}}| < |\mathcal{Q} - \mathcal{Q}_0'| + |\mathcal{Q}_0' - \overline{\mathcal{Q}_0}| + \overline{\mathcal{Q}_0} - \mathcal{Q}'| + |\mathcal{Q}' - \bar{\mathcal{Q}}| < \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon$.

Finally, $\tau(\cdot, v) = \rho(\cdot, \sigma(\cdot, v))$, so

$$
\begin{aligned}
\|\tau(\cdot, v) - v\|_\infty &\le \|\rho(\cdot, \sigma(\cdot, v)) - \sigma(\cdot, v)\|_\infty + \|\sigma(\cdot, v) - v\|_\infty \\
&\le \varepsilon/4(\|\sigma(\cdot, v)\|_\infty + \|v\|) \le \varepsilon/4(\|\sigma(\cdot, v) - v\|_\infty + 2v) \\
&< \varepsilon\|v\|;
\end{aligned}
$$

a similar calculation, of course, is valid for τ^{-1}. □

Proof of Theorem 6.1 from Lemma 6.2. First, by Theorem 5.1, we choose a reparametrization ϕ_{τ_0} of ϕ with $\|\tau_0(\cdot, v) - v\|_\infty$, $\|\tau_0^{-1}(\cdot, v) - v\| < \varepsilon_0\|v\|$ (where ε_0 is chosen less than $\varepsilon/4$), and a partition $\mathcal{Q}_0$ with $(\phi_{\tau_0}, \mathcal{Q}_0) \approx (\psi, \mathcal{Q})$. Next choose finite subsets $V_1 \subset V_2 \subset \cdots \subset \mathbf{R}^n$ with $\bigcup_j V_j$ dense. Choose

$\varepsilon_1 < \varepsilon/16$ and a reparametrizing map σ_1 for ϕ_{τ_0} by Lemma 6.2 and $\mathcal{Q}_1$, so that $\|\sigma_1(\cdot, v) - v\|$, $\|\sigma_1^{-1}(\cdot, v) - v\| < \varepsilon_1\|v\|$ and $|\mathcal{Q}_1 - \mathcal{Q}_0| < \varepsilon_1$; choose W_1 finite so that if $\tau_1 = \sigma_1 \circ \tau_0$, then $\bigvee_{v\in W_1} \phi_v^{(1)-1}(\mathcal{Q}_1) \supset^{\varepsilon_1} \mathcal{P}$ and $(\phi^{(1)}, \mathcal{Q}_1) \approx (\psi, \mathcal{Q})$, where $\phi^{(1)} = \phi_{\tau_1} = (\phi_{\tau_0})_{\sigma_1}$.

We carry things one step further. Choose $\varepsilon_2 < \varepsilon/64$. Choose $\overline{V}_1$ finite, containing V_1, so that $\mathcal{P}_{\overline{V}_1} \supset^{\varepsilon_2/10} \mathcal{Q}_1$. Then, by Lemma 6.2, there is a reparametrizing map σ_2 for ϕ_{τ_1}, and a partition $\mathcal{Q}_2$, such that $\|\sigma_2(\cdot, v) - v\|_\infty$, $\|\sigma_2^{-1}(\cdot, v) \cdot v\| < \varepsilon_2\|v\|$, $|\mathcal{Q}_2 - \mathcal{Q}_1| < \varepsilon_1$, and for some finite $W_2 \subset \mathbf{R}^n$, setting $\tau_2 = \sigma_2 \circ \tau_1$ and $\phi^{(2)} = \phi_{\tau_2}$, we have $\bigvee_{v\in W_2} \phi_v^{(2)-1}(\mathcal{Q}_2) \supset^{\varepsilon_2} \mathcal{P}_{\overline{V}_1}$ and $(\phi^{(2)}, \mathcal{Q}_2) \approx (\psi, \mathcal{Q})$. If ε_2 is chosen small enough, we will further have $\bigvee_{v\in W_1} \phi_v^{(1)-1}(\mathcal{Q}_2) \supset^{(1+1/2)\varepsilon_1} \mathcal{P}$.

Now the general step. Suppose we have $\overline{V}_1, \overline{V}_2, \dots, \overline{V}_{N-1} \subset \mathbf{R}^n$ with $\overline{V}_j \supset V_j$, positive numbers $\varepsilon_j < \varepsilon/4^j$, finite sets $W_1, \dots, W_N$, successive reparametrizing maps $\tau_j = \sigma_j \circ \tau_{j-1}$, and partitions $\mathcal{Q}_0, \dots, \mathcal{Q}_N$, so that: (a) $\|\sigma_j(\cdot, v) - v\|_\infty$ and $\|\sigma_j^{-1}(\cdot, v) - v\| < \varepsilon_j\|v\|$, (b) $|\mathcal{Q}_j - \mathcal{Q}_{j-1}| < \varepsilon_j$, and (c) setting $\phi^{(j)} = \phi_{\tau_j}$, we have, setting $a(j) = 1 + \cdots + 2^{-j}$,

$$\bigvee_{v\in W_j} \phi_v^{(j)^{-1}}(\mathcal{Q}_k) \supset^{a(k-j)\varepsilon_j} \mathcal{P}_{\overline{V}_{j-1}} \quad \text{if } 1 < j < k < N.$$

Choose $\varepsilon_{N+1} < \varepsilon/4^{N+1}$. Choose $\overline{V}_N \supset V_N$ so that $\mathcal{P}_{\overline{V}_N} \supset^{\varepsilon_{N+1}/10} \mathcal{Q}_N$. Apply Lemma 6.2 to get a reparametrizing map σ_{N+1} for $\phi^{(N)}$ and $\mathcal{Q}_{N+1}$ so that $\|\sigma_{N+1}(\cdot, v) - v\|_\infty$, $\|\sigma_{N+1}^{-1}(\cdot, v) - v\|_\infty < \varepsilon_{N+1}\|v\|$,

$$|\mathcal{Q}_{N+1} - \mathcal{Q}_N| < \varepsilon_{N+1}, \qquad \bigvee_{v\in\mathbf{R}^n} \phi_v^{(N+1)^{-1}}(\mathcal{Q}_{N+1}) \supset^{\varepsilon_{N+1}} \mathcal{P}_{\overline{V}_N},$$

and $(\phi^{(N+1)}, \mathcal{Q}_{N+1}) \approx (\psi, \mathcal{Q})$, where $\phi^{(N+1)} = \phi_{\tau_{N+1}} = (\phi_{\tau_N})_{\sigma_{N+1}}$. Choose W_{N+1} finite so that

$$\bigvee_{v\in W_{N+1}} \phi_v^{(N+1)^{-1}}(\mathcal{Q}_{N+1}) \supset^{\varepsilon_{N+1}} \mathcal{P}_{\overline{V}_N}.$$

If ε_{N+1} is small enough, we will also have that

$$\bigvee_{v\in W_j} \phi_v^{(k)-1}(\mathcal{Q}_{N+1}) \supset^{a(N+1-j)\varepsilon_j} \mathcal{P}_{\overline{V}_{j-1}},$$

if $1 < j < k < N$. Thus the induction has proceeded another step.

Let $\overline{\mathcal{Q}} = \lim_{N\to\infty} \mathcal{Q}_N$. Since

$$\bigvee_{v\in W_j} \phi_v^{(k)^{-1}}(\mathcal{Q}_{N+1}) \supset^{2\varepsilon_j} \mathcal{P}_{\overline{V}_{j-1}}$$

for all $N > k > j > 1$, we also have

$$\bigvee_{v\in W_j} \phi_v^{(k)^{-1}}(\overline{\mathcal{Q}}) \supset^{2\varepsilon_j} \mathcal{P}_{\overline{V}_{j-1}} \quad \text{if } k > j > 1.$$

Next, an argument exactly like the one in the proof of Theorem 5.1 shows that if the ε_j are chosen small enough, then τ_j converges to some τ and τ_j^{-1} to τ^{-1}, and the desired uniform Lipschitz conditions hold.

Now

$$\bigvee_{v\in W_j} \phi_v^{(k)^{-1}}(\bar{\mathcal{Q}}) \supset^{2\varepsilon_j} \mathcal{P}_{\bar{V}_{j-1}} \quad \text{if } k > j.$$

Because of the Lebesgue theorem on continuity of translation, we have, setting $\bar{\phi} = \phi_\tau$,

$$\bigvee_{v\in W_j} \bar{\phi}_v^{-1}(\bar{\mathcal{Q}}) \supset^{2\varepsilon_j} \mathcal{P}_{\bar{V}_{j-1}} \quad \text{for all } j.$$

Then

$$\bigvee_{v\in \mathbf{R}^n} \bar{\phi}_v^{-1}(\bar{\mathcal{Q}}) \supset^{2\varepsilon_j} \mathcal{P}_{\bar{V}_{j-1}} \quad \text{for all } j,$$

and so $\bigvee_{v\in\mathbf{R}^n} \bar{\phi}_v^{-1}(\bar{\mathcal{Q}}) \supset \mathcal{P}_{\mathbf{R}^n}$. Since $(\phi^{(j)}, \mathcal{Q}_j) \approx (\psi, \mathcal{Q})$ and $|\mathcal{Q}_j - \bar{\mathcal{Q}}| < \varepsilon_{j+1} + \varepsilon_{j+2} + \dots$, we have $\bar{d}((\phi^{(j)}, \bar{\mathcal{Q}}), (\psi, \mathcal{Q})) < \varepsilon_{j+1} + \varepsilon_{j+2} + \dots$; consequently, again by the Lebesgue theorem on continuity of translations, we have $\bar{d}((\bar{\phi}, \bar{\mathcal{Q}}), (\psi, \mathcal{Q})) = 0$ or $(\bar{\phi}, \bar{\mathcal{Q}}) \approx (\psi, \mathcal{Q})$. □

7. An example of a non-LK n-flow of zero entropy. Recall the construction of the nonloosely Bernoulli process $(T, \mathcal{P})$ of Feldman [F1]. There are $N(0)$ 0-blocks $\{a_{0,i}\}_{i=1}^{N(0)}$, forming the partition $\mathcal{P}$, and if we are given the list $\{a_{m,i}\}_{i=1}^{N(m)}$ of m-blocks, the list $\{a_{m+1,i}\}_{i=1}^{N(m+1)}$ of $(m+1)$-blocks is defined as follows:

$$a_{m+1,1} = \left(a_{m,1}^{N(m)^2} \cdots a_{m,N(m)}^{N(m)^2}\right)^{N(m)^{2N(m+1)}},$$

$$a_{m+1,2} = \left(a_{m,1}^{N(m)^4} \cdots a_{m,N(m)}^{N(m)^4}\right)^{N(m)^{2(N(m+1)-1)}},$$

$$\vdots$$

$$a_{m+1,N(m+1)} = \left(a_{m,1}^{N(m)^{2N(m+1)}} \cdots a_{m,N(m)}^{N(m)^{2N(m+1)}}\right)^{N(m)^2}.$$

A process is built such that every name divides into m-blocks for every m. It will be convenient for us to take $N(m) = 200 \cdot 2^m$.

If we suspend $(T, \mathcal{P})$, we get a continuous one-dimensional process of zero entropy; we call this continuous process $(T_t, \mathcal{P})$. Let $(\phi, \mathcal{P})$ be the product of $(T_t, \mathcal{P})$ with a trivial continuous $(n-1)$-dimensional process. The names in $(\phi, \mathcal{P})$ thus vary only in the direction of the first coordinate, and they split up into m-blocks as above; but now each 0-block occupies some slab of width one in the orbit instead of a single point.

We claim that $(\phi, \mathcal{P})$ is not LK. Let (X, μ) be the underlying space of $(\phi, \mathcal{P})$, and let $L(m) =$ the width of an m-block divided by 2. Then if $x \in X$, the $(\mathcal{P}, C_{L(m)})$ name of x is made up of the terminal segment of some m-block $B_{m,t}(x)$ followed by the initial segment of some other m-block $B_{m,i}(x)$. If $E \subset X$ with $\mu(E) > \frac{3}{4}$, and m is an integer, then there exist x and y in E such that $B_{m,t}(x) \neq B_{m,t}(y)$, $B_{m,i}(y)$ and $B_{m,i}(x) \neq B_{m,t}(y)$, $B_{m,i}(y)$. If we had

$f_{L(m)}(x, y) < \varepsilon$, then some $h \in \mathfrak{D}_{C_{L(m)}}$ would have to match some segment in the $C_{L(m)}$ name of x parallel to the first coordinate axis better than ε; at least one fourth of such a segment lies entirely within some m-block and has image under h lying within some different m-block. If we assume $\|h' - I\|_\infty < \frac{1}{2}$, then this image will have extent at least $\frac{1}{4}L(m)$ in the direction of the first coordinate; because the value of the process $(\phi, \mathcal{P})$ depends only upon the origin and the first coordinate, we may assume that this image is again a line segment parallel to the first coordinate axis. We have thus reduced the problem to the one-dimensional.

7.1. LEMMA. *Let α and β be segments of length at least $\frac{1}{4}L(m)$ lying within different one-dimensional m-blocks, on intervals I_α and I_β. If h: $I_\alpha \to I_\beta$ is piecewise differentiable with $\|h' - 1\|_\infty$, $\|(h')^{-1} - 1\|_\infty < \frac{1}{400}$, then $d(\alpha \circ h^{-1}, \beta) > \frac{1}{2}$.*

PROOF. The proof is by induction on m. Suppose we know that for some m, if $\bar{\alpha}$, $\bar{\beta}$, and $\bar{h}$ are as above, then $d(\bar{\alpha} \circ \bar{h}^{-1}, \bar{\beta}) > \delta_m$ (if $m = 0$, we may take $\delta_m = 1$). Let $\alpha \subset a_{m+1,j}$, $\beta \subset a_{m+1,k}$ $(j < k)$, I_α, I_β, and h be as above.

The segment β has the form $A_{(t)}A_1 \cdots A_t A_{(i)}$, where $A_i = a_{m,c}^{N(m)^{2j}}$ for some c and $A_{(t)}$ and $A_{(i)}$ are terminal and initial segments of certain A_i's. The segment α has the form $\bar{A}_{(t)}\bar{A}_1^d \cdots \bar{A}_s^d A_{(i)}$ where $\bar{A}_i$ has the same form as A_i, $d = N(m)^{2(k-j)}$, and $\bar{A}_{(t)}$ and $\bar{A}_{(i)}$ are terminal and initial segments of certain $\bar{A}_i^d$'s. Restrict h to the subintervals of I_α corresponding to the various $\bar{A}_i^d$'s. The image in β under h of such a subinterval has the form $A_{(t)}A_1 \cdots A_r A_{(i)}$. Keep h the same on the middle $(1 - 2N(m)^{-1})d$ copies of $\bar{A}_i$, but redefine h to be linear on the extreme $2N(m)^{-1}d$ copies so that the image of $\bar{A}_i^d$ under h now has the form $A_1 \cdots A_r A_{r+1}$. Where h has been redefined, we have that $|h' - 1|$, $|(h')^{-1} - 1| < \frac{1}{400}$. Call the new map h_1. We have that $d(\alpha \circ \bar{h}_1^{-1}, \beta) < d(\alpha \circ h^{-1}, \beta) + 3(1 + \frac{1}{400})N(m)^{-1}$. Now restrict h_1^{-1} to the various A_i's in the range of h_1. $h_1^{-1}(A_i)$ has the form $b_{(t)}b_1 \cdots b_r b_{(i)}$, where b_i is an m-block and $b_{(t)}$, $b_{(i)}$ terminal and initial segments of certain m-blocks. We redefine h_1^{-1} in the same way as we redefined h; keep h_1^{-1} the same on the middle $(1 - 2N(m)^{-1})N(m)^{2j}$ copies of the m-block making up A_i and redefine it linearly on the extreme $2N(m)^{2j-1}$ copies of the m-block so that the image becomes $b_1 \cdots b_r b_{r+1}$. Where h_1^{-1} has been redefined, we have $|(h_1^{-1})' - 1|$, $|h_1' - 1| < \frac{1}{400}$. Call the new map h_2^{-1}. We have that

$$\begin{aligned} d\big(\alpha \circ h_2^{-1}, \beta\big) &< d\big(\alpha \circ h_1^{-1}, \beta\big) + 3N(m)^{-1} \\ &< d\big(\alpha \circ h_1^{-1}, \beta\big) + 3\big(1 + \tfrac{1}{400}\big)N(m)^{-1} + 3N(M)^{-1} \\ &< d(\alpha \circ h, \beta) + 7N(m)^{-1}. \end{aligned}$$

Now, h_2 maps $\bar{A}_i^d$ to an interval of almost the same length. Thus, since it occurs with frequency less than $5N(m)^{-1}$, one of the m-blocks comprising $\bar{A}_i^d$

has more than one fourth of it mapped by h_2 to the same m-block in β. Therefore, $d(\alpha \circ h_2^{-1}, \beta) > (1 - 5N(m)^{-1})\delta_m$ and so

$$d(\alpha \circ h^{-1}, \beta) > (1 - 5N(m)^{-1})\delta_m - 7N(m)^{-1} > \delta_m - 12N(m)^{-1} = \delta_{m+1}.$$

Since $\lim_{m\to\infty} \delta_m = 1 - \frac{12}{400}$, we are finished. $\square$

Bibliography

[A] P. Abramov, *Entropy of induced transformations*, Dokl. Akad. Nauk SSSR **128** (1959), 647–650. (Russian)

[D] H. Dye, *On groups of measure-preserving transformations*. I, Amer. J. Math. **81** (1959), 119–159.

[F1] J. Feldman, *New K-automorphisms and a problem of Kakutani*, Israel J. Math. **24** (1976), 16–37.

[F2] ———, *r-entropy, equipartition, and Ornstein's isomorphism theorem in* $\mathbf{R}^n$ (to appear).

[K] S. Kakutani, *Induced measure-preserving transformation*, Proc. Imp. Acad. Tokyo **19** (1943), 635–641.

[Ka] A. Katok, *Time change, monotone equivalence, and standard dynamical systems*, Dokl. Akad. Nauk SSSR **273** (1975), 789–792. (Russian)

[L] D. Lind, *Locally compact measure-preserving flows*, Advances in Math. **15** (1975), 175–193.

[N] D. Nadler, *Abramov's formula for reparametrizations of n-flows* (to appear).

[O] D. Ornstein, *Randomness, ergodic theory, and dynamical systems*, Yale Mathematical Monographs, No. 5.

[P-S] C. Pugh and M. Shub, *Ergodic elements of ergodic actions*, Compositio Math. **23** (1971), 115–122.

[R1] D. Rudolph, *Nonequivalence of measure-preserving transformations*, notes.

[R2] ———, *A dye theorem for n-flows*, $n > 1$ (unpublished).

[R3] ———, *An integrably Lipschitz reparametrization of an n-flow is isomorphic to some tempered reparametrization of the same n-flow* (unpublished).

[Sa] E. Satayev, *An invariant of monotone equivalence which determines families of automorphisms which are monotone equivalent to a Bernoulli automorphism*, Proc. Fourth Sympos. on Information Theory, Part I, Moscow-Leningrad, 1976. (Russian)

[S] Ja. Sinai, *A weak isomorphism of transformations with an invariant measure*, Dokl. Akad. Nauk SSSR **147** (1962), 797–800. (Russian)

[W] B. Weiss, *Equivalence of measure preserving transformations*, Lecture Notes, Hebrew University of Jerusalem.

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